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AN ALGORISM FOR DIFFERENTIAL INVARIANT THEORY.

By Oliver E. Glenn.

It is my purpose to formulate in this paper, for the theory of differential invariants as derived by transformation of binary differential quantics, an algorism of fundamental simplicity* which I have described for algebraic concomitants in research papers written heretofore. Briefly stated the methods relate to certain irrational expressions in the arbitrary functions occurring in the coefficients of the transformations, which serve to define a domain of rationality R within which all differential invariants previously known are functions of certain elementary invariants, in R, and their derivatives, together with arbitrary functions and their derivatives. These elementary invariants, to be designated, in the present paper at least, as *invariant elements*, have served to unify and to simplify to an appreciable degree algebraic theories such as those of boolean† and orthogonal‡ concomitants, and in fact also that of the general algebraic concomitants, as, by their use, I developed a new proof of Gordan's theorem§ which is at least as simple as any other known proof of this important finiteness theorem.||

1. Differential forms. Suppose that two quadratic differential forms

$$\begin{aligned} f &= a dx_1^2 + 2b dx_1 dx_2 + c dx_2^2, \\ f' &= A dy_1^2 + 2B dy_1 dy_2 + C dy_2^2, \end{aligned}$$

in which a, b, c are functions of x_1, x_2 and A, B, C are functions of y_1, y_2, can be so related by an arbitrary functional connection between the variables, that is,

$$x_1 = x_1(y_1, y_2), \qquad x_2 = x_2(y_1, y_2), \tag{1}$$

that when (1) is substituted in f it becomes f'. This implies transformation of the variables and also the differentials, the latter by the substitutions

$$T : dx_i = \frac{\partial x_i}{\partial y_1} dy_1 + \frac{\partial x_i}{\partial y_2} dy_2 \quad (i = 1, 2). \tag{2}$$

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* Lemoine, "Considérations générales sur la mesure de la simplicité dans les sciences mathématiques, etc." Mathematical Papers, International Math. Congr., Chicago, 1893.

† Boole, Cambr. Math. Journ., vol. 3 (1843), p. 1.

‡ Elliott, Proc. Lond. Math. Soc., vol. 33 (1901), p. 226.

§ Cayley, Coll. Math. Papers, vol. 2, p. 250; Gordan, Journ. für Math., vol. 69 (1868), p. 323.

|| O. E. Glenn, Trans. Amer. Math. Soc., vol. 20 (1919), p. 203.

Under these conditions

$$
\begin{aligned}
A &= a\left(\frac{\partial x_1}{\partial y_1}\right)^2 + 2b\,\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_1} + c\left(\frac{\partial x_2}{\partial y_1}\right)^2,\\
B &= a\,\frac{\partial x_1}{\partial y_1}\frac{\partial x_1}{\partial y_2} + b\left(\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} + \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}\right) + c\,\frac{\partial x_2}{\partial y_1}\frac{\partial x_2}{\partial y_2},\\
C &= a\left(\frac{\partial x_1}{\partial y_2}\right)^2 + 2b\,\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_2} + c\left(\frac{\partial x_2}{\partial y_2}\right)^2.
\end{aligned}
\tag{3}
$$

These three differential equations of the first order, if solved, would give the transformations (1).

We next write the general form F of order m under the notation

$$F = \sum_{r=0}^{m} \binom{m}{r} a_r dx_1^{m-r} dx_2^{r}$$

and express it as a symbolical mth power, employing equivalent symbols f, φ, $\cdots$, and writing $\partial f/\partial x_1 = f_1$, $\partial f/\partial x_2 = f_2$, f being, symbolically, a function of x_1, x_2. Thus we may write

$$F = (f_1 dx_1 + f_2 dx_2)^m = (df)^m = (d\varphi)^m = \cdots,$$

whence

$$a_r = f_1^{m-r} f_2^{r} \quad (r = 0, \cdots, m). \tag{3_1}$$

2. **The domain** $R(1, T, \Delta)$. The poles of the transformations (2) in the differentials dx_1, dx_2 are the zeros of the linear forms

$$h_{\pm 1} = 2\,\frac{\partial x_2}{\partial y_1}\,dx_1 + \left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} \pm \Delta\right) dx_2,$$

where

$$\Delta = \left[\left(\frac{\partial y_2}{\partial x_2} - \frac{\partial x_1}{\partial y_1}\right)^2 + 4\,\frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}\right]^{\frac{1}{2}},$$

and two functions $f_{\pm 1}$ which satisfy the equalities

$$
\begin{aligned}
df_{+1} &= \frac{\partial f_{+1}}{\partial x_1}\,dx_1 + \frac{\partial f_{+1}}{\partial x_2}\,dx_2 \equiv h_{+1},\\
df_{-1} &= \frac{\partial f_{-1}}{\partial x_1}\,dx_1 + \frac{\partial f_{-1}}{\partial x_2}\,dx_2 \equiv h_{-1},
\end{aligned}
\tag{4}
$$

will evidently be such that their functional determinant is equal to $4\Delta\, \partial x_2/\partial y_1$. Thus we may write $df_{\pm 1} = h_{\pm 1}$ provided $\Delta \neq 0$; the condition that the substitutions T be non-parabolic, and provided $\partial x_2/\partial y_1 \neq 0$, or that the functions x_1, x_2 be independent and contain both variables y_1, y_2 explicitly. Thus these considerations in connection with the poles of T give a transformation on the differentials whose coefficients appertain to a domain $R(1, T, \Delta)$, the notation for which indicates that the functions therein are expressions with numerical coefficients in the functions, and partial derivatives thereof, occurring in the transformations (1) and (2), in the functional coefficients of F and their derivatives and so forth; and in Δ.

Besides this it is important to observe that integration of the equations $df_{\pm 1} = h_{\pm 1}$ would give x_1, x_2 as functions of f_{+1}, f_{-1} so that the sets x_1, x_2; y_1, y_2; f_{+1}, f_{-1} are functionally interrelated.

The quantics $df_{\pm 1}$ are formally invariantive under T; the multipliers in the invariant relations however are powers of the two factors, in the domain $R(1, T, \Delta)$, of

$$D = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}\,.$$

That is, $df_{\pm 1}$ are differential covariants the invariant relations for which are

$$(5_1) \qquad df_{+1}' = \rho_{+1}{}^{-1} df_{+1}, \qquad df_{-1}' = \rho_{-1}{}^{-1} df_{-1} \equiv \rho_{+1} D^{-1} df_{-1},$$

primes indicating functions of y_1, y_2, and where

$$\rho_{\pm 1} = \frac{1}{2}\left(\frac{\partial x_1}{\partial y_1} + \frac{\partial x_2}{\partial y_2} \pm \Delta\right), \qquad \rho_{+1}\rho_{-1} = D.$$

3. **The invariant elements.** We employ henceforth the following abbreviations:

$$\frac{\partial x_1}{\partial y_1} = \alpha_1, \qquad \frac{\partial x_1}{\partial y_2} = \alpha_2, \qquad \frac{\partial x_2}{\partial y_1} = \beta_0, \qquad \frac{\partial x_2}{\partial y_2} = \beta_1,$$

whence the inverse of the transformations (4) takes the form

$$T'\colon \begin{cases} dx_1 = (-\,4\beta_0\Delta)^{-1}[(\gamma_1 - \Delta)df_{+1} - (\gamma_1 + \Delta)df_{-1}], \\ dx_2 = (-\,4\beta_0\Delta)^{-1}[2\beta_0(-\,df_{+1} + df_{-1})] \quad (\gamma_1 = \beta_1 - \alpha_1). \end{cases}$$

The substitution T' operated upon the form F gives a unique expansion of the latter whose symbolical expression is

$$(5) \quad F = (-\,4\beta_0\Delta)^{-m}\{[(\gamma_1 - \Delta)f_1 - 2\beta_0 f_2]df_{+1} \\ + [-\,(\gamma_1 + \Delta)f_1 + 2\beta_0 f_2]df_{-1}\}^m.$$

Hence

$$(6) \qquad F = \sum_{i=0}^{m}\binom{m}{i}\varphi_{m-2i}\, df_{+1}{}^{m-i} df_{-1,i},$$

in which

$$(7) \quad \varphi_{m-2i} = [(\gamma_1 - \Delta)f_1 - 2\beta_0 f_2]^{m-i}[-\,(\gamma_1 + \Delta)f_1 + 2\beta_0 f_2]^i \\ \times (-\,4\beta_0\Delta)^{-m} \quad (i = 0, \cdots, m).$$

THEOREM I. *The functions φ_{m-2i}, which belong to the domain $R(1, T, \Delta)$, are differential invariants.*

Transformation of F by (1), (2) gives $F' = F$, and expansion of F in the arguments df_{+1}, df_{-1} yields the formula (6). We can also expand F' in the covariant arguments

$$df_{\pm 1}' = 2\frac{\partial x_2}{\partial y_1}dy_1 + \left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} \pm \Delta\right)dy_2,$$

with the result

$$F' = \sum_{i=0}^{m}\binom{m}{i}\varphi_{m-2i}'\, df_{+1}'^{\,m-i}\, df_{-1}'^{\,i}. \tag{8}$$

Hence, by substitution from (5_1), we obtain, after equating coefficients of like powers of $df_{\pm 1}$, the following invariant relations for φ_{m-2i}:

$$\varphi_{m-2i}' = \rho_{+1}^{\,m-2i}D^i\varphi_{m-2i} \quad (i = 0, \cdots, m). \tag{9}$$

We can prove, accordingly, a quite general theorem on the reducibility of a differential invariant. Transformation T' can be written

$$T': \begin{cases} dx_1 = \dfrac{\partial x_1}{\partial f_{+1}}df_{+1} + \dfrac{\partial x_1}{\partial f_{-1}}df_{-1}, \\[2ex] dx_2 = \dfrac{\partial x_2}{\partial f_{+1}}df_{+1} + \dfrac{\partial x_2}{\partial f_{-1}}df_{-1}, \end{cases}$$

and is as general as the transformation T. Let B be any differential invariant whatsoever of F under T for which $B' = \alpha B$, that is,

$$B(a_r';\, dy_1,\, dy_2) = \alpha B(a_r;\, dx_1,\, dx_2). \tag{10}$$

Then a cognate relation holds when the transformed form is (6), viz.,

$$B(\varphi_{m-2r};\, df_{+1},\, df_{-1}) = \alpha B(a_r;\, dx_1,\, dx_2).$$

It is desirable however to state this result in more general terms. Note first that any arbitrary function u of a_r $(r = 0, \cdots, m)$, dx_1, dx_2, by virtue merely of its being arbitrary, satisfies the relation $u' = u$ under the transformation T. Let Ω be a function of the functions a_r and of their x_1, x_2 derivatives and dx_1, dx_2, and also of a certain number of other specified functions u_1, u_2, $\cdots$ for which a relation $u' = u$ holds, among which functions some which are arbitrary functions may be comprised, together with the x_1, x_2 derivatives of u_1, u_2, $\cdots$. Let Ω' be the same function of a_r' and the y_1, y_2 derivatives of a_r' $(r = 0, \cdots, m)$ and of dy_1, dy_2, together with u_1', u_2', $\cdots$ and their y_1, y_2 derivatives. Then, if $\Omega' = \alpha\Omega$, Ω is called a differential parameter. In particular when there are no arbitrary functions actually involved in Ω it is a differential invariant. Hence,

THEOREM II. *Every differential parameter Ω is reducible in the domain*

$R(1, T, \Delta)$, in terms of the $m + 1$ invariant elements φ_{m-2i} $(i = 0, \cdots, m)$ and their x_1, x_2 derivatives, together with arbitrary functions and the differential covariants df_{+1}, df_{-1}.

Note that all of the elements in terms of which Ω is thus reducible are invariantive with the exception of the derivatives of φ_{m-2i} $(i = 0, \cdots, m)$, and we shall prove that these derivatives, also, satisfy an extended form of invariant relation.

The exact form in which an invariant appears as a function of invariant elements will be illustrated by means of two well-known differential parameters, viz.,

$$\begin{aligned}\Delta_1 u &= a_2 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_1} + 2a_1 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} + a_0 \frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_2} \\ &= \varphi_{-2}\left(\frac{\partial u}{\partial f_{+1}}\right)^2 + 2\varphi_0 \frac{\partial u}{\partial f_{+1}}\frac{\partial u}{\partial f_{-1}} + \varphi_2\left(\frac{\partial u}{\partial f_{-1}}\right)^2, \\ \nabla(u, v) &= a_2 \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + a_1\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1}\right) + a_0 \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2} \\ &= \varphi_{-2}\frac{\partial u}{\partial f_{+1}}\frac{\partial v}{\partial f_{+1}} + \varphi_0\left(\frac{\partial u}{\partial f_{+1}}\frac{\partial v}{\partial f_{-1}} + \frac{\partial u}{\partial f_{-1}}\frac{\partial v}{\partial f_{+1}}\right) + \varphi_2 \frac{\partial u}{\partial f_{-1}}\frac{\partial v}{\partial f_{-1}}.\end{aligned}$$

These appertain to the case $m = 2$. A pure invariant for this case is the discriminant $\delta = a_1^2 - a_0 a_2$, and

$$\delta = 4\beta_0^2\Delta^2(\varphi_0^2 - 4\varphi_2\varphi_{-2}).$$

4. Types of parameters. The algorism established by the preceding theorems affords a classification of differential parameters into types some of which consist of invariants of what seem to be entirely new categories.

(*a*) *The formal type.* The quantic F is formally analogous to a binary algebraical quantic, the transformations (2) to the linear transformations of such a quantic employed in algebraic invariant theory and the relations (3) and their generalizations to the transformations of the induced group in that theory. There is therefore a type of concomitant and a theory appertaining thereto closely analogous to formal algebraic invariant theory, the concomitants being expressible rationally in terms of symbols* $f_1, f_2, \varphi_1, \varphi_2, \cdots; dx_1, dx_2$, and reducible in terms of invariant elements $\varphi_m, \cdots, \varphi_{-m}$; df_{+1}, df_{-1} in the manner illustrated in the example of the discriminant of the quadratic in the preceding section. There are differential identities for the reduction of such concomitants which can be cast in a formal mould so as to be, in effect, similar to the symbolism of

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* Maschke, Trans. Amer. Math. Soc., vol. 1 (1900), p. 197, and ibid., vol. 4 (1903), p. 445. Haskins, Trans. Amer. Math. Soc., vol. 3 (1902), p. 71. A. W. Smith, Trans. Amer. Math. Soc., vol. 7 (1906), p. 33. Ricci and Levi-Civita, Math. Annalen, vol. 54 (1901).

Aronhold and Clebsch. Thus when we abbreviate as follows:

$$\frac{\partial U}{\partial x_i} = U_i, \qquad \frac{\partial^2 U}{\partial x_i \partial x_k} = U_{ik},$$

$$U_1 V_2 - U_2 V_1 = (U, V), \tag{11}$$

and assume $m = 2$, we get

$$\alpha'(F, \varphi)' = \alpha(F, \varphi) \quad (\alpha = 1/\sqrt{\delta}),$$

where F, φ are any parameters, and

$$\Delta_1 u = \alpha^2 (F, u)^2,$$

$$\nabla(u, v) = \alpha^2 (F, u)(F, v),$$

$$(a, b)(c, d) + (a, c)(d, b) + (a, d)(b, c) = 0.$$

(*b*) *The extended formal type.* From (3_1) there follows

$$\frac{\partial a_r}{\partial x_1} = ((m-r) f_{11} f_2 + r f_{21} f_1) f_1^{m-r-1} f_2^{r-1},$$

$$\frac{\partial a_r}{\partial x_2} = ((m-r) f_{12} f_2 + r f_{22} f_1) f_1^{m-r-1} f_2^{r-1} \quad (r = 0, \cdots, m).$$

Hence differential parameters involving the first derivatives of the functions a_r are represented symbolically by means of expressions constructed from the combinations below and generalizations to higher derivatives will be obvious:

$$f_1^{m-r-1} f_2^r f_{11}, \qquad \varphi_1^{m-r-1} \varphi_2^r \varphi_{11}, \qquad f_1^{m-r} f_2^{r-1} f_{21}, \qquad f_1^{m-r-1} f_2^r f_{12},$$

and so forth.

An example* is the following for the quadratic form which we write under the notation

$$F = \sum_{i,k=1}^{2} a_{ik} dx_i dx_k \quad (a_{ki} = a_{ik}):$$

$$\Delta_2 u = \alpha(F, \alpha(F, u)) = \sum_{r,s} A_{rs} \frac{\partial^2 u}{\partial x_r \partial x_s} - \sum_{r,s,i,k} A_{rs} A_{ik} \begin{bmatrix} rs \\ k \end{bmatrix} \frac{\partial u}{\partial x_k},$$

where $\begin{bmatrix} rs \\ k \end{bmatrix}$ is the so-called triple index symbol due to Christoffel;

$$\begin{bmatrix} rs \\ k \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_{rk}}{\partial x_s} + \frac{\partial a_{sk}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_k}\right) = f_k f_{rs},$$

and A_{rs} denotes the minor of a_{rs} in $\delta = |a_{rs}|$. The expression in terms of the symbols is

$$\Delta_2 u = \alpha^2 [f_1(f_{12} u_2 - f_{22} u_1) - f_2(f_{11} u_2 - f_{12} u_1)] + \cdots.$$

The symbolism of the paragraphs (*a*), (*b*) was discovered by Maschke. Expression of $\Delta_2 u$ in terms of invariant elements is obtained by forming its invariant relation, according to theorem II.

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* Beltrami's second differential parameter. Compare J. E. Wright, Invariants of Quadratic Differential Forms (Cambridge Tracts, 1908).

(*c*) *The orthogonal type.* I shall designate as the orthogonal type of differential parameter those which can be generated in totality by forming rational expressions in invariant elements φ_{m-2i} $(i = 0, \cdots, m)$, df_{+1}, df_{-1} which simplify by multiplication into functions appertaining to the domain $R(1, T, 0)$. The essential forms from which to construct this totality are evidently $P \pm Q$ where P is of the type

$$P = \varphi_m{}^{x_0}\varphi_{m-2}{}^{x_1}\cdots\varphi_{-m}{}^{x_m}df_{+1}{}^{\sigma_1}df_{-1}{}^{\sigma_2}, \tag{12}$$

and Q is the conjugate of P,

$$Q = \varphi_{-m}{}^{x_0}\varphi_{-(m-2)}{}^{x_1}\cdots\varphi_m{}^{x_m}df_{-1}{}^{\sigma_1}df_{+1}{}^{\sigma_2}. \tag{13}$$

An example for the quadratic quantic, $F = a_0dx_1{}^2 + 2a_1dx_1dx_2 + a_2dx_2{}^2$, is the following:

$$\begin{aligned}\Delta(\varphi_2 df_{+1}{}^2 - \varphi_{-2}df_{-1}{}^2) \\ = \left[a_0\frac{\partial x_1}{\partial y_1} - a_0\frac{\partial x_2}{\partial y_2} + 2a_1\frac{\partial x_2}{\partial y_1}\right]dx_1{}^2 + \left[2a_0\frac{\partial x_1}{\partial y_2} + 2a_2\frac{\partial x_2}{\partial y_1}\right]dx_1dx_2 \\ + \left[2a_1\frac{\partial x_1}{\partial y_2} + a_2\frac{\partial x_2}{\partial y_2} - a_2\frac{\partial x_1}{\partial y_1}\right]dx_2{}^2.\end{aligned}$$

(*d*) *The extended orthogonal type.* The x_1, x_2 derivatives of an invariant element φ_{m-2i} are not invariantive. In the quadratic case for instance, or generally, we can obtain the relations which replace invariant relations for $\partial\varphi_{m-2i}/\partial x_j$ $(j = 1, 2)$ by applying to the members of the relation (9) the operators

$$\frac{\partial}{\partial y_1} = \frac{\partial x_1}{\partial y_1}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial y_1}\frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial y_2} = \frac{\partial x_1}{\partial y_2}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial y_2}\frac{\partial}{\partial x_2}. \tag{14}$$

Notwithstanding this fact certain rational combinations of invariant elements, arbitrary functions, *and their derivatives* will belong to the domain $R(1, T, 0)$ and be invariantive. Such expressions will be called differential parameters of the extended orthogonal type.

(*e*) *Unclassified parameters.* Certain invariants, such as some irrational expressions in invariants of the above types, will not belong to any of the above categories and such as do not we leave unclassified.

5. Parameters of the orthogonal type. The invariant systems determined in this section are derived by means of the irreducible solutions of a certain linear diophantine equation and the generality of the conceptions involved is such that the same equation and the analogous algorism concerning invariant elements yields the corresponding systems for binary algebraic forms under orthogonal substitutions (the orthogonal invariants proper), under boolean transformations, the transformations of Einstein (invariants of relativity) and the general systems in $R(1, T, 0)$, all of which I have treated previously having published enumerations for the orders from one to five inclusive. Hence I treat in detail in this paper the parameters of the orthogonal type for the differential quantic of order six.

We are concerned with concomitant expressions in the invariant elements which belong to $R(1, T, 0)$.

To construct such an expression from products such as P in (12) it is necessary, though not sufficient, that the exponent of ρ_{+1} in the invariant relation for P:

$$P' = \rho_{+1}{}^{a} D^{b} P, \tag{15}$$

should be zero (compare (9)). That is,

$$a = \sum_{i=0}^{m} (m - 2i)x_i - \sigma_1 + \sigma_2 = 0. \tag{16}$$

The concomitants P, Q do not belong to $R(1, T, 0)$, but they are in correspondence with $P + Q$, $P - Q$ which, when deprived of irrelevant factors, do appertain to that domain. Hence, according to Hilbert's lemma, the finite set of irreducible solutions of (16) furnish the exponents $x_0, \cdots, x_m, \sigma_1, \sigma_2$ of a complete system in the domain $R(1, T, 0)$. The system for the quantic

$$F = \sum_{r=0}^{6} \binom{6}{r} a_r dx_1{}^{6-r} dx_2{}^{r}$$

is obtained, therefore, by solving the equation

$$6x_0 + 4x_1 + 2x_2 + \sigma_2 = \sigma_1 + 2x_4 + 4x_5 + 6x_6, \tag{17}$$

and if an irreducible solution giving a product P is $(x_0, x_1, x_2, x_4, x_5, x_6, \sigma_1, \sigma_2)$ the solution which gives the conjugate product Q is $(x_6, x_5, x_4, x_2, x_1, x_0, \sigma_2, \sigma_1)$. The table below furnishes the irreducible solutions, in conjugate pairs, under the suggestive notation $\alpha_{\pm 1} = P \pm Q$. Thus, for example, a quadratic covariant is

$$\lambda_{\pm 1} = \varphi_6 \varphi_{-4}{}^2 df_{-1}{}^2 \pm \varphi_{-6} \varphi_4{}^2 df_{+1}{}^2.$$

To solve (17) I wrote it in the form

$$6x + 4y + 2z + w = 0, \tag{18}$$

$$x = x_0 - x_6, \qquad y = x_1 - x_5, \qquad z = x_2 - x_4, \qquad w = \sigma_2 - \sigma_1. \tag{19}$$

An appropriate number of solutions of (18) in both positive and negative integers being determined, the sets for (17) are furnished by solving (19). Thus the problem is subdivided into a large number of mutually exclusive subproblems. Including the invariant φ_0 not furnished by (17) and the covariant $\eta = df_{+1} df_{-1}$, the number of concomitants in the system is 31. There are 14 invariants ($\sigma_1 = \sigma_2 = 0$) and 17 covariants.

	x_0	x_1	x_2	x_4	x_5	x_6	σ_1	σ_2
α	1					1		
β		1			1			
γ			1	1				
$\delta \pm 1$		1		2				
			2		1			
$\epsilon \pm 1$	1			3				
			3			1		
$\zeta \pm 1$	1			1	1			
		1	1			1		
$\theta \pm 1$	1		1		2			
		2		1		1		
$\iota \pm 1$	2				3			
		3				2		
$\kappa \pm 1$			1				2	
				1				2
$\lambda \pm 1$		2				1	2	
	1				2			2
$\mu \pm 1$	1				1		2	
		1				1		2
$\nu \pm 1$	1			2			2	
			2			1		2
$\varphi \pm 1$	1			1			4	
			1			1		4
$\rho \pm 1$		1					4	
					1			4
$\sigma \pm 1$		1		1			2	
			1		1			2
$\tau \pm 1$	1						6	
						1		6

6. Parameters of the extended orthogonal type. From (8) it follows, by correspondence, that if $B(a_r;\ dx_1,\ dx_2,\ \cdots)$ is any differential parameter of F there exists a relation

$$B(\varphi_{m-2r}';\ df_{+1}',\ df_{-1}',\ \cdots) = QB(\varphi_{m-2r};\ df_{+1},\ df_{-1},\ \cdots).$$

We next prove certain results concerning the derivatives of invariant elements. From equations (14),

$$\frac{\partial}{\partial y_1} = \alpha_1 \frac{\partial}{\partial x_1} + \beta_0 \frac{\partial}{\partial x_2}, \qquad \frac{\partial}{\partial y_2} = \alpha_2 \frac{\partial}{\partial x_1} + \beta_1 \frac{\partial}{\partial x_2}, \tag{20}$$

and hence one obtains, by differentiation of the relation

$$\varphi_{m-2i}' = \rho_{+1}{}^{m-2i} D^i \varphi_{m-2i} = Q^{(i)} \varphi_{m-2i} \quad (i = 0, \cdots, m), \tag{21}$$

the following formulas:

$$\frac{\partial^r \varphi_{m-2i}'}{\partial y_1{}^{r-s} \partial y_2{}^s} = \alpha_1{}^{r-s} \beta_1{}^s Q^{(i)} \frac{\partial^r \varphi_{m-2i}}{\partial x_1{}^{r-s} \partial x_2{}^s} + \delta_r{}^{(i)} \tag{22}$$

$$(r = 0, 1, 2, \cdots; \quad s = 0, 1, \cdots, r; \quad i = 0, 1, \cdots, m),$$

where

$$\delta_r{}^{(i)} = \frac{\partial^{r-1} \varphi_{m-2i}}{\partial x_1{}^{r-s-1} \partial x_2{}^s} \left(\alpha_1 \frac{\partial}{\partial x_1} + \beta_0 \frac{\partial}{\partial x_2} \right) \left(\alpha_1{}^{r-s-1} \beta_1{}^s Q^{(i)} + \delta_{r-1}{}^{(i)} \right) + \alpha_1{}^{r-s-1} \beta_0 \beta_1{}^s Q^{(i)} \frac{\partial^r \varphi_{m-2i}}{\partial x_1{}^{r-s-1} \partial x_2{}^{s+1}} .$$

Thus while the derivatives of the invariant elements are not invariantive they satisfy equations which are obtained by adding increments $\delta_r{}^{(i)}$ to what would be, except for the increments, invariant relations of regular type for these derivatives. Hence we obtain

THEOREM III. *In order that a homogeneous function of the coefficients* a_r *and their derivatives,*

$$B(a_r;\ \partial^t a_r / \partial x_1{}^{t-u} \partial x_2{}^u;\ dx_1, dx_2),$$

should be a concomitant, it is necessary that the increment to

$$B(\varphi_{m-2r};\ \partial^t \varphi_{m-2r} / \partial x_1{}^{t-u} \partial x_2{}^u;\ df_{+1} df_{-1}), \tag{23}$$

under the appropriate equations (22), *should be zero. Subject to evident conditions of isobarism, this condition is also sufficient.*

When $r = 1$ we derive from the relations (22), which then reduce to

$$\frac{\partial \varphi_{m-2i}'}{\partial y_2} = \alpha_2 Q^{(i)} \frac{\partial \varphi_{m-2i}}{\partial x_1} + \beta_1 Q^{(i)} \frac{\partial \varphi_{m-2i}}{\partial x_2} + (\alpha_2 Q_{x_1}{}^{(i)} + \beta_1 Q_{x_2}{}^{(i)}) \varphi_{m-2i},$$

$$\frac{\partial \varphi_{m-2i}'}{\partial y_1} = \alpha_1 Q^{(i)} \frac{\partial \varphi_{m-2i}}{\partial x_1} + \beta_0 Q^{(i)} \frac{\partial \varphi_{m-2i}}{\partial x_2} + (\alpha_1 Q_{x_1}{}^{(i)} + \beta_0 Q_{x_2}{}^{(i)}) \varphi_{m-2i},$$

the formulae of transvection (cf. (11)),

$$\begin{aligned} \psi_{m-2i}' &= (\varphi_{m-2i}', Q^{(i)\prime}) = Q^{(i)} D(\varphi_{m-2i}, Q^{(i)}) \\ &\qquad = \rho_{+1}{}^{m-2i} D^{i+1}(\varphi_{m-2i}, Q^{(i)}) \\ &= Q^{(i)} D \psi_{m-2i} = Q_1{}^{(i)} \psi_{m-2i} \quad (i = 0, \cdots, m), \end{aligned} \tag{24}$$

and this establishes the following theorem:

THEOREM IV. *The transvectants ψ_{m-2i}, which belong to the domain $R(1, T, \Delta)$, are relative differential parameters involving first order derivatives of invariant elements φ_{m-2i} $(i = 0, \cdots, m)$, and consequently also first order derivatives of coefficients a_r of the ground form F.*

The relation (24) expressing the invariancy of ψ_{m-2i} is formally identical, save for the replacement of i by $i + 1$ in the exponent of D, with the corresponding relation for φ_{m-2i}. We can therefore form relations in the derivatives of the parameters ψ_{m-2i} similar to those employed in the derivation of the transvectants (24) and thus describe a process of iteration whose equivalent, by formula, is

$$((\varphi_{m-2i}', Q^{(i)\prime}), Q_1^{(i)\prime}) = \rho_{+1}{}^{m-2i} D^{i+2}((\varphi_{m-2i}, Q^{(i)}), Q_1^{(i)}). \tag{25}$$

Thus

$$\psi_{m-2i}{}^{(2)} = ((\varphi_{m-2i}, Q^{(i)}), Q^{(i)})$$

is a differential parameter involving derivatives of the second degree of invariant elements, and the extension to parameters of the rth iteration, functions of the rth derivatives of the invariant elements, is evident. Upon the basis of these iterated transvectants we construct a theory of parameters of the extended orthogonal type.

The invariant $\psi_{-(m-2i)}$ is conjugate to ψ_{m-2i}; hence systems of parameters in $R(1, T, 0)$, of the extended orthogonal type, involving, as to derivatives, powers of first derivatives, only, of invariant elements, can be formed upon the basis of combinations $P \pm Q$, where

$$P = \varphi_m{}^{x_0} \varphi_{m-2}{}^{x_1} \cdots \varphi_{-m}{}^{x_m} \psi_m{}^{y_0} \psi_{m-2}{}^{y_1} \cdots \psi_{-m}{}^{y_m} df_{+1}{}^{\sigma_1} df_{-1}{}^{\sigma_2}$$

and Q is the product conjugate to P. That is, in accordance with the lemma of Hilbert, a complete system is given by forms $\alpha_{\pm 1} = P \pm Q$ obtained from the totality of sets of irreducible solutions of the linear diophantine equation

$$\sum_{r=0}^{m}(m - 2r)x_r + \sum_{s=0}^{m}(m - 2s)y_s - \sigma_1 + \sigma_2 = 0. \tag{26}$$

The complete system for the differential quadratic is obtained, therefore, from the irreducible solutions of the equation

$$2x_0 + 2y_0 + \sigma_2 = 2x_2 + 2y_2 + \sigma_1,$$

shown in the table below:

	x_0	x_2	y_0	y_2	σ_1	σ_2
n					1	1
o	1	1				
p			1	1		
$q_{\pm 1}$	1			1		
		1	1			
$r_{\pm 1}$			1		2	
				1		2
$s_{\pm 1}$		1				2
	1				2	

The system consists of 11 forms, viz., φ_0, ψ_0 and

$$
\begin{aligned}
o &= \varphi_2\varphi_{-2}, \qquad p = (\varphi_2, \rho_{+1}{}^2)(\varphi_{-2}, \rho_{-1}{}^2), \\
q_{\pm 1} &= \varphi_2(\varphi_{-2}, \rho_{-1}{}^2) \pm \varphi_{-2}(\varphi_2, \rho_{+1}{}^2), \\
r_{\pm 1} &= (\varphi_2, \rho_{+1}{}^2)df_{+1}{}^2 \pm (\varphi_{-2}, \rho_{-1}{}^2)df_{-1}{}^2, \\
s_{\pm 1} &= \varphi_2 df_{+1}{}^2 \pm \varphi_{-2}df_{-1}{}^2, \qquad \eta = df_{+1}df_{-1}.
\end{aligned}
\tag{27}
$$

Note the equality of the indices, i.e., the power of D appearing as a multiplier in the invariant relations for the terms of each binomial quantic in the list. Furthermore, the corresponding index for P is equal to that for Q but we can prove a still more general result. Let the first iteration of the transvectant $\psi_{m-2i} = \psi_{m-2i}{}^{(1)}$ in relation (25) be designated by $\psi_{m-2i}{}^{(2)}$:

$$\psi_{m-2i}{}^{(2)} = ((\varphi_{m-2i}, Q^{(i)}), Q_1{}^{(i)}) \quad (i = 0, \cdots, m),$$

and let the kth iterated transvectant be $\psi_{m-2i}{}^{(k+1)}$ so that $\varphi_{m-2i} = \psi_{m-2i}{}^{(0)}$. Then the conjugate to $\psi_{m-2i}{}^{(k+1)}$ is $\psi_{-(m-2i)}{}^{(k+1)}$ and the concomitant Q which is conjugate to

$$P = \prod_{i=0}^{m}\prod_{k=0}^{n} \psi_{m-2i}{}^{(k)x_{ik}} df_{+1}{}^{\sigma_1} df_{-1}{}^{\sigma_2} \cdot \tag{28}$$

is obtained by the corresponding changes of sign of subscripts.

THEOREM V. *The quantics $\alpha_{\pm 1} = P \pm Q$ are differential invariants of the extended orthogonal type appertaining to the domain $R(1, T, 0)$. They involve derivatives of invariant elements, and therefore of the functional coefficients a_r of the ground-form F itself, of all orders from zero to the nth and constitute an infinitude of quantics which possesses the property of finiteness. A complete system is given by the finite set of irreducible solutions in positive integers of the linear diophantine equation*

$$a = \sum_{k=0}^{n} \sum_{i=0}^{m} (m - 2i)x_{ik} + \sigma_2 - \sigma_1 = 0. \tag{29}$$

We need only the proof that the index for P equals that of Q. These indices are, respectively,

$$\begin{aligned} \alpha &= \sum_{k=0}^{n} \sum_{i=0}^{m} (i + k)x_{ik} - \sigma_2, \\ \beta &= \sum_{k=0}^{n} \sum_{i=0}^{m} (m + k - i)x_{ik} - \sigma_1. \end{aligned} \tag{30}$$

But $\beta - \alpha = a = 0$; hence $\alpha = \beta$.

I remark in conclusion that properly chosen polynomials in the expressions characteristic of parameters of the extended orthogonal type, no assumption here being made that these are generally expressible in any particular form except as polynomials in invariant elements and the derivatives of these, will be, when multiplied out, expressions which are free from the functions involved in the transformations, that is, parameters which belong to the domain $R(1, 0, 0)$. These are the concomitants to which the investigations of former writers relate. It is apparent that a finiteness theorem can be stated for the parameters in the latter domain and treated in the way analogous to that exemplified in my proof of the theorem of Gordan, quoted previously in this paper, and these and other developments would probably be of sufficient importance to warrant detailed treatment.

Differential parameters of the orthogonal types, containing arbitrary functions other than those involved in the transformations (1) and (2), are obtained from the relations

$$D\frac{\partial}{\partial x_2} = \alpha_1\frac{\partial}{\partial y_2} + \alpha_2\left(-\frac{\partial}{\partial y_1}\right), \qquad D\left(-\frac{\partial}{\partial x_1}\right) = \beta_0\frac{\partial}{\partial y_2} + \beta_1\left(-\frac{\partial}{\partial y_1}\right). \tag{31}$$

If $B(a_r;\ \partial^s a_r/\partial x_1^{s-t}\partial x_2^t;\ dx_1,\ dx_2)$ is any covariantive parameter, then, a parameter involving an additional arbitrary function u is

$$B(a_r;\ \partial^s a_r/\partial x_1^{s-t}\partial x_2^t;\ \partial/\partial x_2,\ -\partial/\partial x_1)u.$$

This method can be employed to derive an infinitude of concomitants involving arbitrary functions u, v, $\cdots$, from the systems derived in preceding sections.

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